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APPROXIMATION OF INTEGRATED SEMIGROUPS

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Abstract. The purpose of this paper is to show an integrated semigroup on a Banach space can be approximated by a sequence of integrated semigroups acting on different Banach spaces.

1. Introduction

The initial value problem in a Banach space X

$$u'(t) = Au(t), \quad u(0) = x$$

has been extensively studied if A is the generator of a C_0 semigroup. Hille-Yosida theorem gives the necessary and sufficient conditions in order that A is the generator of a C_0 semigroup [4]. One of these conditions is the density of the domain of A. But there are many examples that is formulated in the above problem without the density of the domain of A (see [3]). In this case the concept of integrated semigroup introduced by Arendt [1] is very useful to treat the above problem.

In this paper we study the approximation of an integrated semigroup on a Banach space X by a sequence of the integrated semigroups on Banach spaces X_n. In order to prove our result, we use Theorem 2.2 in [5] that the convergence of the sequence of functions $\{f_n : [0,\ \infty) \to X\}$ is equivalent to the convergence of their Laplace transforms and the equicontinuity of $\{f_n\}$.

Let X and X_n be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_n$, $n = 1, 2, \cdots$, respectively. For each n, there exist bounded linear operators $P_n : X \to X_n$ and $E_n : X_n \to X$ satisfying

(i) $\|P_n\| \leq M_1$ and $\|E_n\|_n \leq M_2$, where M_1 and M_2 are independent of n.

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(ii) $\lim_{n\to\infty} \|E_n P_n x - x\| = 0$ for every $x \in X$.
(iii) $P_n E_n = I_n$, where I_n is the identity operator on X_n.

In general we do not have $X_n \subset X$. If one has numerical approximation in mind, then the spaces X_n are finite dimensional.

Throughout this paper, X is a Banach space and $B(X)$ is the space of all bounded linear operators from X to X. For a linear operator A, we denote the domain, the range, the resolvent set and the resolvent by $D(A)$, $Ran(A)$, $\rho(A)$ and $R(\lambda,\ A)$, respectively.

2. Approximation

First we recall the definition of integrated semigroups.

Definition 2.1. A linear operator A on a Banach space X is called the generator of an integrated semigroup if there exist constants M, $\omega \geq 0$ and a strongly continuous function $S : [0,\ \infty) \to B(X)$ with $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ such that $(\omega,\ \infty) \subset \rho(A)$ and $R(\lambda,\ A)x = \lambda \int_0^\infty e^{-\lambda t} S(t)xdt$ for $\lambda > \omega$ and $x \in X$.

In this case, $\{S(t)\}_{t\geq 0}$ is called the integrated semigroup generated by A.

It is known in [2] that a closed linear operator A in X is the generator of a locally Lipschitz continuous integrated semigroup on X if and only if there exist constants $M,\ \omega \geq 0$ such that

$$(\omega,\ \infty) \subset \rho(A) \quad \text{and} \quad \|(\lambda I - A)^{-k}\| \leq \frac{M}{(\lambda-\omega)^k}$$

for $\lambda > \omega$ and $k \geq 1$, and every locally Lipschitz continuous integrated semigroup is exponentially bounded.

Main result of this paper is given by the following theorem.

Theorem 2.2. *Let A be the generator of an integrated semigroup $\{S(t)\}_{t\geq 0}$ on X satisfying $\|S(t)\| \leq Me^{\omega t}$ for some constants $M,\ \omega \geq 0$ and all $t \geq 0$. Let $\{T_n\}$ be a sequence of linear operators with $T_n \in B(X_n)$ and let $\{h_n\}$ be a positive null sequence with the following properties.*

(i) $\|T_n\|_n \leq Me^{\omega k h_n}$ *for $k \geq 0$ and $n \geq 1$.*
(ii) *For $x \in D(A)$ there exists a sequence $\{x_n\}$ with $x_n \in X_n$ such that $\lim_{n\to\infty} E_n x_n = x$ and $\lim_{n\to\infty} E_n A_n x_n = Ax$, where $A_n = (T_n - I_n)/h_n$.*

Then

$$\lim_{n\to\infty}\int_0^t E_n T_n^{[s/h_n]} P_n x ds = S(t)x \quad \textit{for } x \in X$$

and the convergence is uniform on bounded $t-$intervals, where $[r]$ is the integer part of $r \geq 0$.

Proof. Since $A_n \in B(X_n)$, A_n is the generator of a uniformly continuous semigroup $\{e^{tA_n}\}_{t\geq 0}$ on X_n and

$$\begin{aligned} \|e^{tA_n}\|_n &\leq e^{-t/h_n}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{t}{h_n}\right)^k \|T_n^k\|_n \\ &\leq Me^{t/h_n(e^{\omega h_n}-1)} \leq Me^{te^{\omega}}. \end{aligned}$$

Choose $a > e^{\omega}$. Then $\|e^{tA_n}\| \leq Me^{at}$ for all $t \geq 0$. By Hille-Yosida theorem, $(a,\ \infty) \subset \rho(A_n)$ and $\|R(\lambda,\ A_n)\| \leq M/(\lambda - a)$ for $\lambda > a$.

For $y \in Ran(\lambda I - A)$, there exists $x \in D(A)$ such that $y = (\lambda I - A)x$. By hypothesis there exist $x_n \in X_n$ such

$$\lim_{n\to\infty} E_n x_n = x \quad \text{and} \quad \lim_{n\to\infty} E_n A_n x_n = Ax.$$

Set $(\lambda I_n - A_n)x_n = y_n$. Then we have

$$\lim_{n\to\infty} E_n y_n = \lim_{n\to\infty} E_n(\lambda I_n - A_n)x_n = (\lambda I - A)x = y.$$

So we have for $\lambda > a$

$$\begin{aligned} &\|E_n R(\lambda,\ A_n)P_n y - R(\lambda,\ A)y\| \\ &\leq \|E_n R(\lambda,\ A_n)P_n y - E_n R(\lambda,\ A_n)y_n\| \\ &\quad + \|E_n R(\lambda,\ A_n)y_n - R(\lambda,\ A)y\| \\ &\leq M_2\|R(\lambda,\ A_n)P_n y - R(\lambda,\ A_n)y_n\|_n + \|E_n x_n - x\| \\ &\leq \frac{M_2 M}{\lambda - a}\|P_n y - y_n\|_n + \|E_n x_n - x\| \\ &= \frac{M_2 M}{\lambda - a}\|P_n y - E_n P_n y_n\|_n + \|E_n x_n - x\| \\ &\leq \frac{M_1 M_2 M}{\lambda - a}\|y - E_n y_n\| + \|E_n x_n - x\| \ \to\ 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

By the density of $Ran(\lambda I - A)$, we have

$$\lim_{n\to\infty} E_n R(\lambda,\ A_n)P_n x = R(\lambda,\ A)x \quad \text{for } x \in X.$$

Let $x \in X$. Then

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t} \int_0^t T_n^{[s/h_n]} P_n x ds dt \\
&= \int_0^\infty \int_s^\infty e^{-\lambda t} T_n^{[s/h_n]} P_n x dt ds \\
&= \frac{1}{\lambda} \int_0^\infty e^{-\lambda s} T_n^{[s/h_n]} P_n x ds \\
&= \frac{1}{\lambda} \sum_{k=0}^\infty \int_{kh_n}^{(k+1)h_n} e^{-\lambda s} T_n^k P_n x ds \\
&= \frac{1 - e^{-\lambda h_n}}{\lambda^2} \sum_{k=0}^\infty e^{-\lambda k h_n} T_n^k P_n x \\
&= \frac{1 - e^{-\lambda h_n}}{\lambda^2} \left(I_n - e^{-\lambda h_n} T_n \right)^{-1} P_n x \\
&= \frac{1 - e^{-\lambda h_n}}{\lambda^2} \left(I_n - e^{-\lambda h_n} (I_n + h_n A_n) \right)^{-1} P_n x \\
&= \frac{1 - e^{-\lambda h_n}}{\lambda^2} \frac{e^{\lambda h_n}}{h_n} \left(\frac{e^{\lambda h_n} - 1}{h_n} I_n - A_n \right)^{-1} P_n x.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\lim_{n \to \infty} \int_0^\infty e^{-\lambda t} \int_0^t E_n T_n^{[s/h_n]} P_n x ds dt \\
&= \lim_{n \to \infty} \frac{1 - e^{-\lambda h_n}}{\lambda^2} \frac{e^{\lambda h_n}}{h_n} E_n \left(\frac{e^{\lambda h_n} - 1}{h_n} I_n - A_n \right)^{-1} P_n x \\
&= \frac{1}{\lambda} (\lambda I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S(t) x dt.
\end{aligned}$$

We have proved that the Laplace transforms of $\int_0^t E_n T_n^{[s/h_n]} P_n x ds$ converge to the Laplace transform of the integrated semigroup $\{S(t)\}_{t \geq 0}$.

Next we will show the equicontinuity of $\{\int_0^t E_n T_n^{[s/h_n]} P_n x ds\}$. For $0 \leq s < t \leq T$,

$$\begin{aligned}
&\| \int_0^t T_n^{[r/h_n]} P_n x dr - \int_0^s T_n^{[r/h_n]} P_n x dr \|_n \\
&= \| \int_s^t T_n^{[r/h_n]} P_n x dr \|_n
\end{aligned}$$

$$
\begin{aligned}
&\le \int_s^t M e^{\omega[r/h_n]h_n} \|P_n x\|_n dr \\
&\le M \int_s^t e^{\omega r} dr \|P_n x\|_n \\
&\le M e^{\omega T} \|P_n\|_n |t-s|.
\end{aligned}
$$

Hence $\{\int_0^t E_n T_n^{[s/h_n]} P_n x ds\}$ is equicontinuous. By Theorem 2.2 in [5] we have the result. $\square$

EXAMPLE 2.3. *Let $X = C([0,\ 1])$ with the supremum norm and let $A : D(A) \subset X \to X$ be a linear operator defined by $Au = -u'$ with $D(A) = \{u \in X : u(0) = 0,\ u' \in X\}$.*

Then the closure of $D(A)$ is $C_0([0,\ 1])$, which is not dense in X. For $\lambda > 0$ and $v \in X$, define

$$
u(t) = \int_0^t e^{-\lambda s} v(t-s) ds, \quad t \in [0,\ 1].
$$

Then $u \in D(A)$, $(\lambda I - A)u = v$ and

$$
\begin{aligned}
|u(t)| \quad &\le \quad \int_0^t e^{-\lambda s} |v(t-s)| ds \\
&\le \quad \|v\| \int_0^t e^{-\lambda s} ds \le \frac{1}{\lambda} \|v\|
\end{aligned}
$$

So $(0,\ \infty) \subset \rho(A)$ and $\|R(\lambda,\ A)\| \le 1/\lambda$, that is, A is a Hille-Yosida operator. By Theorem 2.4 in [2], A is the generator of an integrated semigroup $\{S(t)\}_{t \ge 0}$.

Let $X_n = R^n$ with the supremum norm. Define $P_n : X \to R^n$ and $E_n : R^n \to X$ by

$$
P_n u = (u(1/n),\ u(2/n),\ \cdots,\ u(n/n)) \text{ and } E_n x^{(n)} = f_n,
$$

where $u \in X$, $x^{(n)} = (x_1^{(n)},\ x_2^{(n)},\ \cdots,\ x_n^{(n)}) \in R^n$ and $f_n(0) = x_1^{(n)}$, $f_n(k/n) = x_k^{(n)}$, $k = 1,\ 2,\ \cdots,\ n$ and linear between two consecutive points. Then $\|P_n\| \le 1$, $\|E_n\| \le 1$, $P_n E_n = I_n$ and $\lim_{n\to\infty} E_n P_n u = u$ for all $u \in X$.

Define $A_n : R^n \to R^n$ by

$$
A_n x^{(n)} = n(-x_1^{(n)},\ x_1^{(n)} - x_2^{(n)},\ \cdots,\ x_{n-1}^{(n)} - x_n^{(n)}).
$$

Then A_n is linear and $\|A_n\|_n \leq 2n$. Let $u \in D(A)$. Then

$$\begin{aligned}
&A_nP_nu \\
&= n(-u(1/n),\ u(1/n) - u(2/n),\ \cdots,\ u((n-1)/n) - u(n/n)) \\
&= -(u'(c_1),\ u'(c_2),\ \cdots,\ u'(c_n))
\end{aligned}$$

for some $c_i \in ((i-1)/n,\ i/n)$, $i = 1,\ 2,\ \cdots,\ n$ and

$$P_nAu = -(u'(1/n),\ u'(2/n),\ \cdots,\ u'(n/n)).$$

Since u' is continuous, $\lim_{n\to\infty} \|A_nP_nu - P_nAu\|_n = 0$. Hence we have

$$\begin{aligned}
&\|E_nA_nP_nu - Au\| \\
&\leq \|E_nA_nP_nu - E_nP_nAu\| + \|E_nP_nAu - Au\| \\
&\leq M_2\|A_nP_nu - P_nAu\|_n + \|E_nP_nAu - Au\| \to 0 \text{ as } \ n \to \infty.
\end{aligned}$$

Choose a sequence $\{h_n\}$ with $0 < h_n < 1/n$. Then

$$\begin{aligned}
T_nx^{(n)} &= x^{(n)} + h_nA_nx^{(n)} \\
&= \Big((1 - nh_n)x_1^{(n)},\ nh_nx_1^{(n)} + (1 - nh_n)x_2^{(n)}, \\
&\qquad \cdots,\ nh_nx_{n-1}^{(n)} + (1 - nh_n)x_n^{(n)}\Big)
\end{aligned}$$

Then $\|T_n\|_n \leq 1$ and so we have

$$\lim_{n\to\infty} \int_0^t E_nT_n^{[s/h_n]}P_nuds = S(t)u \text{ for } \ u \in X.$$

That is, the values computed by the difference equations converge to the integrated semigroup.

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